JHEP HYPER VERSION

Received: *April 27, 2000*, Accepted: *July 23, 2000*

D2-branes in B fields

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Abstract: This note focuses on the coupling of a type-IIA D2-brane to a background B field. It is shown that the D0-brane charge arising from the integral over the D2-brane of the pullback of the B field is cancelled by bulk contributions, for a compact D2-brane wrapping a homotopically trivial cycle in space-time. In M-theory this cancellation is a straightforward consequence of momentum conservation. This result resolves a puzzle recently posed by Bachas, Douglas and Schweigert related to the quantization of R-R charges on stable spherical D2-branes on the group manifold SU(2).

Keywords: M-Theory, D-branes.

JHEP07(2000)039

Contents

1. Introduction

A D2-brane in a general type-IIA supergravity background has a world-volume action given by a sum of Born-Infeld and Wess-Zumino terms [1, 2, 3]

$$S = S_{\rm BI} + S_{\rm WZ}\,.$$

The Born-Infeld part of the action is given by [4]

$$S_{\rm BI} = -T_2 \int e^{-\Phi}\sqrt{-\det(G_{\alpha\beta} + \mathcal{F}_{\alpha\beta})}\,,$$

where $G_{\alpha\beta}$ is the pullback of the space-time metric to the D2-brane world-volume, Φ is the pullback of the dilaton and and

$$\mathcal{F}_{\alpha\beta} = 2\pi\alpha' F_{\alpha\beta} - B_{\alpha\beta}$$

combines the pullback of the space-time B field and the field strength F of the U(1) gauge field living on the brane.

The Wess-Zumino terms couple the brane to the space-time R-R fields through [5, 6]

$$S_{\rm WZ} = \int \left(\sum_i C^{(i)}\right) \wedge e^{\mathcal{F}}\,.$$

In particular there is a term in the D2-brane action of the form

$$\int C^{(1)} \wedge (2\pi\alpha' F - B)\,. \tag{1.1}$$

This is the term which we will discuss in this note. In particular, we will focus on the coupling between the space-time R-R 1-form field $C^{(1)}$ and the B field which is mediated by the D2-brane in this interaction term.

For a compact D2-brane of arbitrary topology, the first Chern class of the U(1) bundle on the brane at a fixed point in time gives an integer

$$\frac{1}{2\pi}\int F = k\,.$$

According to (1.1), this integer acts as a source for the R-R vector field and represents k D0-branes bound to the D2-brane [6]. This interpretation of the U(1) flux on the D2-brane also naturally follows from T-duality [7, 8]. The quantization of $\int F$ corresponds nicely with our expectation that the number of D-particles in any system is integral.

The coupling of the R-R vector field to the pullback of the space-time B field on the D2-brane is at first sight somewhat more surprising, as there is no natural reason for the integral $\int B$ to be quantized. Indeed, in a region of constant $H = dB$ flux one can imagine blowing up a small spherical D2-brane out of the vacuum, which would have a continuously varying value of $\int B$. This lack of quantization of $\int B$ was found by Bachas, Douglas and Schweigert to be particularly puzzling in the context of D-branes on the SU(2) group manifold, where stable spherical D2-branes are predicted by conformal field theory [9]. These stable spherical D2-branes would seem from (1.1) to have non-integral, and in fact irrational values of D0-brane charge.

In this note we clear up this puzzle. We show that for a compact D2-brane embedded on a homotopically trivial spatial cycle the D0-brane charge associated with $\int B$ is precisely cancelled by a contribution from the bulk fields. This cancellation is guaranteed by conservation of D0-brane charge in the full supergravity + D2-brane theory. From the point of view of M-theory, this cancellation can be seen in terms of conservation of momentum in the compact direction.

In section 2 we give a brief discussion of an analogous situation in 4D classical electromagnetism which should make the physical argument quite transparent. In section 3 we repeat this analysis in the context of M-theory, and in section 4 we interpret the discussion in terms of the IIA language. Section 5 contains a discussion of the connection with other work and the resolution of the puzzle posed by Bachas, Douglas and Schweigert. After this note was written, we learned that a similar resolution of this puzzle has been found by Polchinski [10].

2. Electromagnetic analogy

Consider classical electromagnetism in 4 dimensions. Let us restrict attention to physical configurations which are invariant under translation in the X^3 direction, so that we may choose a gauge in which the vector potential A_μ is independent of this coordinate. Let us assume that there is a background magnetic field $F_{i3} = \partial_i A_3$ present, for $i = 1, 2$.

Now, let us separate a pair of equal and opposite charges $+q, -q$ by starting with both charges at a point $\mathbf{a}$ and moving the charge $+q$ along a path $\mathcal{P}$ in the 1-2 plane to the point $\mathbf{b}$. This gives us an electric dipole. In the process of separating the charges we move the positive charge through the magnetic field F_{i3}. The Lorentz force $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$ gives a net momentum in the X^3 direction

$$\begin{aligned} p_3 &= -q \int_{\rm a}^{\rm b} F_{i3} \, dx^i \\ &= q \left(A_3(\mathbf{a}) - A_3(\mathbf{b}) \right) \end{aligned} \tag{2.1}$$

to the electric dipole. Note that this momentum is independent of the path $\mathcal{P}$ since $dF = 0$. By conservation of momentum, this momentum must be cancelled by the momentum in the electromagnetic fields. Indeed, we can compute the momentum in the Poynting vector flux $\mathbf{E} \times \mathbf{B}$, of which the X^3 component is

$$\begin{aligned} \int F_{0i} F^{i3} &= \int F_{0i} \, \partial_i A_3 \\ &= -\int A_3 \, \partial_i F_{0i} \\ &= \int A_3 \, q(\delta(\mathbf{x} - \mathbf{b}) - \delta(\mathbf{x} - \mathbf{a})) \\ &= q \left(A_3(\mathbf{b}) - A_3(\mathbf{a}) \right), \end{aligned}$$

which precisely cancels (2.1).

This calculation is easily generalized to show that any static configuration of charges with net charge 0 in the presence of a magnetic field F_{i3} has a total momentum contained in the electromagnetic fields whose component in the X^3 direction is equal and opposite to the momentum of the charges when the charges each are taken to have total momentum

$$p_3 = -q A_3 \,.$$

This calculation is precisely analogous to the result for a membrane moving in a background 4-form field strength $F_{\mu\nu\lambda\,11}$ in M-theory, which we now discuss.

3. M-theory picture

The low-energy description of M-theory is given by 11-dimensional supergravity. In addition to the metric tensor and gravitino field, 11D supergravity has a dynamical 3-form field C_{IJK} which is closely analogous to the U(1) vector field A_μ of 4D electromagnetism. M-theory contains dynamical membranes which couple electrically to the 3-form field through a term of the form

$$\int_V d^3\xi^\alpha \, C \,,$$

where V is the membrane world-volume and C is the pullback of the 3-form to V. The curvature of C is a 4-form $F = dC$ with components

$$F_{IJKL} = 4\,\partial_{[I}C_{JKL]}\,.$$

Let us consider a compact membrane Σ of arbitrary genus embedded in a space of topology $\mathbb{R}^{10}$, in the presence of a magnetic field strength $F_{\mu\nu\lambda\,11}$. Although there may be forces such as the membrane tension acting on the membrane we can imagine that the membrane is held in a static position by some additional external forces. With the application to type-IIA D2-branes in mind, we will imagine that the field configuration is independent of X^{11} so that all components of C_{IJK} can be chosen to be independent of X^{11}. Since our membrane is homotopically trivial, we can imagine starting with the entire membrane at a fixed point a in space and expanding the membrane to its desired shape. We can parameterize this family of deformations of the membrane with a parameter $\tau \in [0,1]$. We denote by Γ the 3-volume swept out by $\Sigma \times [0,1]$ as we vary τ. In performing the expansion of the membrane from a point to the desired geometry $\hat{\Sigma} = \partial\Gamma$, we must move the membrane in the background magnetic field, which imparts to it a net momentum in the X^{11} direction

$$\begin{aligned} p_{11} &= -\frac{1}{6}\int_{\Gamma} F_{ijk\,11}\, dX^i \wedge dX^j \wedge dX^k \\ &= -\frac{1}{2}\int_{\hat{\Sigma}} C_{ij\,11}\, dX^i \wedge dX^j\,. \end{aligned} \tag{3.1}$$

This is the analogue in M-theory of the momentum arising from the force $q\mathbf{v} \times \mathbf{B}$ in classical electromagnetism. Note that this quantity is invariant under (X^{11}-independent) gauge transformations of the form $\delta C_{ij\,11} = \partial_i\Lambda_j - \partial_j\Lambda_i$ when the membrane is wrapped on a homotopically trivial cycle since $dd\Lambda = 0$.

Because momentum in M-theory is conserved, the momentum imparted to the membrane in this process must be balanced by a momentum flux in the 4-form field strength. Indeed, there is a contribution to the 11-momentum in M-theory from the term in the stress tensor

$$\frac{1}{6}\,F_{0ijk}F^{ijk\,11}\,.$$

Integrating by parts, we can rewrite this contribution to the momentum just as in the electromagnetic analogue by

$$\begin{aligned} \frac{1}{6}\int F_{0ijk}F^{ijk\,11} &= \int F_{0ijk}\;\partial_{[i}C_{jk]\,11} \\ &= -\int C_{[jk\,11}\;\partial_{i]}F_{0ijk} \\ &= \frac{1}{2}\int_{\hat{\Sigma}} C_{ij\,11}\, dX^i \wedge dX^j. \end{aligned} \tag{3.2}$$

This precisely cancels the momentum of the membrane (3.1) given by its motion in the magnetic field.

Thus, we see that it is natural to associate with an M-theory membrane in a magnetic field an intrinsic momentum

$$-\frac{1}{2}\int_{\Sigma} C_{ij\,11}\, dX^i \wedge dX^j \tag{3.3}$$

which is precisely cancelled by the momentum in the 4-form field produced by the interaction between the "electric" field F_{0ijk} produced by the membrane and the external "magnetic" field $F_{jkl\,11}$ in which the membrane is sitting.

4. IIA picture

We would now like to translate the preceding discussion into the language of type-IIA string theory. When 11-dimensional supergravity is dimensionally reduced by compactifying on a small circle in the X^{11} direction, the resulting theory is type-IIA supergravity. Under this dimensional reduction, the 3-form field C_{IJK} decomposes into the R-R 3-form field $C^{(3)}_{\mu\nu\lambda}$ and the NS-NS 2-form field $B_{\mu\nu}$ of the IIA theory. The R-R 1-form field $C^{(1)}_{\mu}$ in the IIA theory arises from the Kaluza-Klein vector field $g_{\mu\,11}$, and the quanta of momentum in the compact direction of M-theory are then associated with D0-branes, which are the objects carrying charges under $C^{(1)}$.

With these identifications, we see that the 11-momentum associated with a membrane in a background magnetic field (3.3) becomes a D0-brane charge on a D2-brane Σ associated with the integrated pullback of the B field

$$-\int_{\Sigma} B\,. \tag{4.1}$$

By the argument described above, this D0-brane charge must be cancelled by an additional contribution to the D0-brane charge associated with fields in the bulk. Indeed, just such a term appears in the action of type-IIA supergravity. The curvatures of the IIA fields $C^{(3)}$ and B are given by

$$\begin{aligned} H &= dB \\ G^{(4)} &= dC^{(3)} + C^{(1)} \wedge H\,. \end{aligned} \tag{4.2}$$

In the IIA supergravity action there is a term quadratic in the curvature $G^{(4)}$

$$-\frac{1}{48}\int d^{10}x\, \sqrt{-g}\, |G^{(4)}|^2\,.$$

From the definition (4.2), we see that this includes a term proportional to

$$(C^{(1)} \wedge dB)\cdot(dC^{(3)})\,. \tag{4.3}$$

This is just the term we need to cancel (4.1), just as the bulk Poynting-type contribution to the 11-momentum (3.2) cancels the membrane momentum (3.3). Indeed, the term contributing to IIA D0-brane charge in (4.3) is precisely the dimensional reduction of (3.2).

To complete the story we can simply check that the D0-brane charge contained in (4.3) indeed can be related through integration by parts to the negative of the D0-brane charge (4.1)

$$\begin{aligned}\frac{1}{6}\int G^{(4)}_{0ijk}H^{ijk} &= \int G^{(4)}_{0ijk}\partial_{[i}B_{jk]} \\ &= -\int B_{[jk}\partial_{i]}G^{(4)}_{0ijk} \\ &= \int_{\Sigma} B\,. \end{aligned} \tag{4.4}$$

Thus, we have shown that the D0-brane charge associated with a D2-brane in an external B field is cancelled by a contribution from the bulk when the D2-brane is embedded on a homotopically trivial cycle. Indeed, by directly using the integration by parts argument in (4.4), we see that this result holds whenever there is no boundary contribution to the integral of the bulk contribution to the D0-brane charge. Note, however, that the interpretation of (4.1) as being the total charge arising from the analogue of the Lorentz force is only valid when the D2-brane can be homotopically contracted to a point.

5. Discussion and examples

We have shown that the term $\int C^{(1)} \wedge B$ in the world-volume action of a D2-brane should be associated with a D0-brane charge which is generally cancelled by an opposite contribution from the bulk fields. In M-theory, this cancellation is a simple consequence of momentum conservation.

There are several situations in which this interpretation of the B field contribution to D0-brane charge on a membrane is useful. For one thing, the cancellation of this contribution to D0-brane charge clarifies the question of quantization of D-particle number in a type-IIA configuration containing D2-branes and B field fluxes. Because the contribution from $\int B$ to the D0-brane charge is always cancelled in the bulk, the quantization of D-particle number is automatically guaranteed by the topological condition that the integral of the U(1) flux $\int F$ is quantized in units of 2π. In particular, this clears up the puzzle posed by Bachas, Douglas and Schweigert in [9]. They found a set of stable spherical D2-branes on the group manifold SU(2), with integrated B fields which seemed to indicate irrational values for D0-brane charge. From the discussion in this note, it is clear that these B field contributions to the D0-brane charge are cancelled by Poynting-type bulk contributions of the form (4.4). Indeed, one could imagine adiabatically moving between any pair of the stable spherical D2-branes found in [9]. In this process, the D2-brane would pick up additional 11-momentum (D-particle charge) from the analogue of the Lorentz force condition, which would be signified by the change in $\int B$. At the same time, the D2-brane would act back on the fields, increasing the net D-particle charge in the bulk from the analogue of the Poynting flux. Thus, we see that the results of [9] are perfectly consistent and that there is no paradox: D0-brane charge is always integrally quantized, and only arises from free D-particles or from the pth Chern class of the U(1) field on a D2p-brane.

Another situation in which the results in this note are relevant is when a D2-brane bubble is produced from a system of N D0-branes through the introduction of a background electric 4-form field. It was shown in [11] that there is a term in the action describing a system of multiple D0-branes in a background 4-form field of the form

$$\mathrm{Tr}\left([X^i, X^j]X^k\right) G^{(4)}_{0ijk} \,. \tag{5.1}$$

The matrix operator in this expression which couples to the background field is simply the dipole moment of the D2-brane charge encoded in the system of D0-branes [12, 11]. It was pointed out by Myers in [13] that in the presence of such a background flux, the lowest energy configuration for a system of multiple D0-branes is a spherical membrane configuration [14] in which the noncommuting matrices X^i are proportional to N-dimensional generators of SU(2). If such a membrane is placed in a nontrivial external B field, according to the mechanism described in this note there should be additional contributions to the D-particle number given by the integral of the B field over the membrane world volume and from the bulk. From (5.1) and the corresponding couplings between the higher moments of the membrane charge and the background described in [11] it is clear that the spherical system of D0-branes correctly acts as a source for the 4-form field, so that there will indeed be an additional bulk contribution to the total D0-brane charge. However, since we know that the net D0-brane charge is really N, there must be an additional term analogous to (4.3) in the nonabelian D0-brane action. In [11], the complete set of linear couplings of a system of multiple D0-branes to supergravity background fields were deduced from matrix theory. The extra term we need here, however, will be a term which couples quadratically to the supergravity background fields. By T-dualizing the 9-brane action as discussed in [15, 13], it is possible to see that such a term indeed appears. Thus, we can see that in the language of D0-branes the results of this note are again reproduced, and that N is indeed the complete D0-brane charge. An example of a situation in which 4-form flux in M-theory is used both to blow up a graviton and to induce (angular) momentum in the resulting membrane is discussed in [16]. Although in this case there is no circle on which M-theory is compactified it would be interesting to understand this picture better, possibly using the mechanism we have discussed here.

In this note we have focused on D2-branes. It is natural to extend this analysis to higher-dimensional Dp-branes. In general, on a D2p-brane there is a coupling of the form $\int C^{(1)} \wedge B \wedge F^{p-1}$. The argument described here extends very easily to this case. We know that F^{p-1} on a D2p-brane corresponds to D2-brane charge, and the coupling in question thus describes precisely the sort of 11-momentum discussed in this note for this D2-brane charge. On any Dp-brane there is also a coupling of the form $\int C^{(p-1)} \wedge B$. These terms are very similar to those discussed here, but do not have the physical interpretation in terms of M-theory momentum; an example of a configuration in which a term of this type is relevant appears in [17]. There are also Wess-Zumino terms in the world-volume action of a Dp-brane which are of quadratic or higher order in the B field. The simplest example of such a term is the term $\int C^{(1)} \wedge B \wedge B$ in the D4-brane action. It would be interesting to see whether terms of this form have a simple interpretation analogous to the discussion in this note.

In this note we have discussed D2-branes which are wrapped on homotopically trivial cycles in space-time. When branes are wrapped on nontrivial cycles, the physics can be more complicated. For example, when D2-branes are wrapped on a nontrivial space-time torus, then the presence of a B field can be interpreted in terms of a world-volume Yang-Mills theory on a noncommutative torus [18]. Recently there have also been discussions of noncommutative geometry in the context of spherical D2-branes in B fields [19, 20, 21]. It would be interesting to understand better how the discussion in this note fits into the framework of noncommutative geometry produced by B fields.

Acknowledgments

I would like to thank S. Das, M. Douglas, I. Ellwood, J. Polchinski and J. Troost for helpful discussions and correspondence. This work was supported in part by the A.P. Sloan Foundation and in part by the DOE through contract #DE-FC02-94ER40818.

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